1 Scaling

In $\mathbb{R}^2$ we can scale a point, $\vec{p}_1 = (x, y)$, to a point, $\vec{p}_2 = (x', y')$, by scale factors, s_x and s_y, as,

$$x' = s_x x \tag{1}$$
$$y' = s_y y \tag{2}$$

This transformation can be written in matrix form as,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

In $\mathbb{R}^{3+1}$ this transformation matrix can be written using homogeneous coordinates as,

$$\mathbf{S} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

2 Translation

In $\mathbb{R}^2$ we can translate a point, $\vec{p}_1 = (x, y)$, to a point, $\vec{p}_2 = (x', y')$, by,

$$x' = x + t_x \tag{5}$$
$$y' = y + t_y \tag{6}$$

Translation is an affine transformation,

$$\vec{x} \mapsto \mathbf{A}\vec{x} + \mathbf{b} \tag{7}$$

We can introduce homogeneous coordinates to make the translation operator linear. Equations 5 and 6 can then be combined in matrix form as,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{8}$$

In $\mathbb{R}^{3+1}$ this operator becomes,

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

3 Rotation

Suppose we have a point, $\vec{p}_1 = (x, y)$, that we want to rotate by an angle, θ_2, to the point, $\vec{p}_2 = (x', y')$. The point, $\vec{p}_1$, can be represented by the polar coordinates (r, θ_1), and we have,

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} && (10) \\
x &= r\cos\theta_1 && (11) \\
y &= r\sin\theta_1 && (12)
\end{aligned}
$$

Similarly, $\vec{p}_2$ can be written in polar coordinate form as $(r, \theta_1 + \theta_2)$, which yields,

$$
\begin{aligned}
x' &= r\cos(\theta_1 + \theta_2) && (13) \\
&= r\cos\theta_1\cos\theta_2 - r\sin\theta_1\sin\theta_2 && (14) \\
&= x\cos\theta_2 - y\sin\theta_2 && (15) \\
y' &= r\sin(\theta_1 + \theta_2) && (16) \\
&= r\cos\theta_1\sin\theta_2 + r\sin\theta_1\cos\theta_2 && (17) \\
&= x\sin\theta_2 + y\cos\theta_2 && (18)
\end{aligned}
$$

using the angle sum identities in equations 14 and 17. This transformation can be written in matrix form as,

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{19}
$$

In $\mathbb{R}^{3+1}$ these linear operators can be written using homogeneous coordinates as,

$$
\mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}
$$

$$
\mathbf{R}_y = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}
$$

$$
\mathbf{R}_z = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}
$$